**1.** Evaluate the binomial coefficient:



Use TI-84 calculator to access the math menu.

Then, go to PROB and press 3 for the nCr (or n choose r).

Enter the appropriate values.

The answer is **1716**.

**2.** A pizza parlor offers a choice of 16 different toppings. How many 2-topping pizzas are possible?

This is a nCr problem. Why? It’s a combination (subset) problem. The order does not matter and repetition is not allowed. You’re supposed to get 2 toppings that are different from each other (it wouldn’t make sense otherwise, I mean, getting olives and olives?). However, if you get olives and tomatoes or tomatoes and olives, it doesn’t really matter.

Use the theorem to get the binomial coefficient 16 choose 2. The answer would be **120**.

**3.** A school dance committee is to consist of 2 freshmen, 3 sophomores, 4 juniors, and 5 seniors. If 5 freshmen, 8 sophomores, 9 juniors, and 8 seniors are eligible to be on the committee, in how many ways can the committee be chosen?

You can determine that repetition is not allowed (can’t choose the same person twice). Order doesn’t matter because it doesn’t matter who you pick, you’re only concerned that the proper amount is chosen from the group. So, this is a combination (subset).

So,

5 choose 2 = 10 freshmen

8 choose 3 = 56 sophomores

9 choose 4 = 126 juniors

8 choose 5 = 56 seniors

Using the Product Principle, you can multiply them all together to get the number of ways the committee can be chosen (), and you get **3951360**.

**4.** A standard deck of cards consist of four suits (clubs, diamonds, hearts, and spades) with each suit containing 13 cards (ace, two through ten, jack, queen, king) for a total of 52 cards in all. How many 5-card hands will consist of exactly 3 kings and 2 queens?

This is a combination. Repetition is not allowed because each card is an individual and you can’t use them again to add to your hand once you pick them. The order doesn’t matter because all you care about is getting 3 kings and 2 queens.

The “5 card hand” doesn’t really have anything to do with solving the problem EXCEPT to let you know there is not an extra card in the hand that would deviate the problem.

So,

4 choose 3 kings = 4

4 choose 2 queens = 6

Multiply them using the Product Principle. = **24**.

**5.** How many different 7-letter permutations can be formed from 5 identical H’s and 2 identical T’s?

This is also a combination problem. Don’t be fooled by the word “permutation”, it just is asking for how many ways a 7 letter “word” can be made. The word permutation is used to tell you that repetition isn’t allowed (at least I think).

Repetition is not allowed because once you choose an H, you will have 4 H’s to choose from when filling spaces. The order doesn’t matter because the important part is getting 5 H’s and 2 T’s.

Think of the 4.4 Labelling Objects and Trinomial Coefficients in the Lecture 1 notes. You can say Subset 1 consists of 5 H’s from an H “bin” and Subset 2 consists of 2 T’s from a “T” bin.

The possibilities for Subset 1 are 7 choose 5. This is basically saying that out of 7 spaces, choose 5 spaces to fill with the 5 H’s. For Subset 2, choose 2 remaining spaces to fill with the 2 T’s. This would appear as:

7 choose 5 = 21

2 choose 2 = 1

Use the Product Principle to multiply them to get **21**.

**6.** How many anagrams can be created from the word “metamorphosis” if the new word does not need to be meaningful?

“Anagram” means a word, phrase, or name formed by rearranging the letters of another.

Like the last problem, it’s also a combination. The order doesn’t matter because the new word doesn’t need to have meaning. Repetition is not allowed because each letter is an individual.

There are 13 letters (or spaces) in the word metamorphosis. So,

13 choose 2 (m) = 78

11 choose 1 (e) = 11

10 choose 1 (t) = 10

9 choose 1 (a) = 9

8 choose 2 (o) = 28

6 choose 1 (r) = 6

5 choose 1 (p) = 5

4 choose 1 (h) = 4

3 choose 2 (s) = 3

1 choose 1 (i) = 1

Multiply them thanks to the Product Principle, and you would get the answer **778377600**.

You can get this in an easier way by counting how many letters are in the word (13), putting a factorial on it, and then noticing that there are doubles of 3 letters (m, o, and s). You can form these as factorials on a denominator too. This is a trinomial coefficient, so you would get .

**7.** Find the coefficient of in .

You can use the Binomial Theorem:



*k* factors of *y*, which would be 7.

This is because the second part of the should resemble the number of subsets (). Since in the problem, there is no *y* (and thus can be replaced with 1, as it is the coefficient of a nonexistent *y* since normally, the *x*’s and *y*’s would be multiplied together).

To figure out the value 7, it should be realized that the exponents of the should add up to the exponent of the parentheses’ expression.

*n* would be the exponent of the parentheses’ expression (15).

*y* would be the coefficient of a nonexistent *y* in this problem (1).

In this problem the exponent

**6435**.

Solved with the help of [this YouTube video](https://www.youtube.com/watch?v=OW96ke3lhno).

**8.** What is the coefficient of in the expansion of ?

Same as the last problem.

Let’s define variables:

*n* = 17, *k* = 10

**-2489344**.

Where does the 2 come from? It’s the coefficient in the original expression of *x*. It needs to be included, as would the coefficient of *y* if it were any value other than 1.

**9.** Expand the expression using the Binomial Theorem:

Basically, you’re finding the coefficients of each option.

For , *n* = 5 and *k* = 0.

**32**

For , *n* = 5 and *k* = 1.

**80**

For , *n* = 5 and *k* = 2.

**80**

For , *n* = 5 and *k* = 3.

**40**

For , *n* = 5 and *k* = 4.

**10**

For no *x,* or when *x* is , *n* = 5 and *k* = 5.

**1**

**10.** A store is selling 5 types of hard candies: cherry, strawberry, orange, lemon, and pineapple. How many ways are there to choose:

1. 24 candies
2. 24 candies with at least a piece of each flavor
3. 24 candies with at least 3 cherry and 4 lemon

Recognize that this problem is a multiset. The question is asking “how many ways”, but what it’s really asking is “how many multisets”.

Part A

So, you use stars and bars to count.

You can choose a random arrangement. Let’s say there are 6 cherry type candies, 4 strawberry type candies, 5 orange type candies, 4 lemon type candies, and 5 pineapple type candies.

You can think of the number of types as stickers; there are 24 pieces of candy, but the exact amount for each is unspecified (remember, you’re creating a sample multiset for this). You’re assigning types to each candy. The candy types are represented by the numbers 1 through 5.

You can write a sample multiset *M* for this:

You can convert this multiset into a stars and bars multiset, represented by *M\**.

This separates the 24 elements (candies) into 5 compartments (based on type). Therefore, this can be viewed as a 28 element list consisting of 24 stars and 4 bars (adding them together). To determine the number of this, you can make a subset *P* to make note of the positions of the stars in *M\** like so (be sure that no numbers repeat in this subset):

Thus, this is a 24-element subset chosen from 28 elements, making a one-to-one correspondence between the multisets *M* and subsets *P*. Since the number of 24-element subsets chosen from 28 elements is given by the binomial coefficient , which is equal to **20475** multisets.

This can be expressed as (()) as a repeated binomial coefficient, which 5 being the pool size (the types of candies, in this case) and 24 being the size of the multiset.

You can also use the formula shown below, where *n* is equal to pool size and *k* is equal to the size of the multiset. If you plug in the appropriate values for this, you’ll get the same answer.



Part B

The condition is that there is 1 of each type. So, you can do to reduce the total number of candies because you already know definitely that 5 candies already have a chosen type. Use the same method as Part A, with the formula, where instead of 24.

**8855** ways.

Part C

Since you know at least 3 are cherry and 4 are lemon for certain, you can do the same as Part B and do to reduce the total number of candies. Use the same method as Part A, with the formula, where instead of 24.

**5985** ways.

**11.** James owns 6 different mathematics books and 6 different computer science books and wishes to fill 5 positions on a shelf. If the first 2 positions are to be occupied by math books and the last 3 by computer books, how many ways can this be done?

It’s important to note that the mathematics and computer science books are different events. For both events, repetition is not allowed because each book is a different one, and when you select one to put on the shelf, you have less options to choose from. For each event, the order doesn’t matter because they aren’t mixed.

Use the formula for each individually as they’re both combinations or basically,

For math books: 30

For computer books: 120

Multiply them together using the Product Principle to get **3600** total ways to order them.

**12.** For the following relations on the set of POSITIVE integers, determine if it satisfies each of the following conditions and enter Y or N in each of the boxes:

|  | **Reflexive** | **Symmetric** | **Transitive** | **Equivalence Relation** |
| --- | --- | --- | --- | --- |
| ***m ~ n* 13 divides *m - n*** | Y | N | N | N |
| ***m ~ n* 6 divides *m + n*** | Y | Y | N | N |
| ***m ~ n* 17 divides *mn*** | Y | Y | N | N |

First One

What do the relations mean?

This is the opposite of divisible. Instead, for example, the first one, let’s say you’re testing for reflexive. You can say *m* and *n* are positive integers 5. So, 5-5 is 0. So, can 13 go into 0 ()? Yes. So it’s reflexive.

To be symmetric, let’s say that 5 and 6 are related to each other. 5 is *m* and 6 is *n*. . Can 13 go into -1? Yes.

**FILL IN LATER**

**13.** For the following relations on the set of college students, determine if it satisfies each of the following conditions and enter Y or N in each of the boxes:

|  | **Reflexive** | **Symmetric** | **Transitive** | **Equivalence Relation** |
| --- | --- | --- | --- | --- |
| ***A ~ B* *A* is shorter than *B*** | N | N | Y | N |
| ***A ~ B* *A, B* took 2 class(es) together** | Y | Y | N | N |
| ***A ~ B* *A, B* have the same major** | Y | Y | Y | Y |

Let’s start with the first row. (Refer to next problem for description of properties.)

It’s NOT reflexive because A is not shorter than A.

It’s NOT symmetric because B is not shorter than A.

For transitive, let’s say that there is a person C. If A is shorter than B, and B is shorter than C, then is C shorter than A? Yes. You can also prove this by assigning heights. A is 4 foot, B is 5 foot, and C is 6 foot. It fits the statements. A is definitely shorter than C.

This is NOT an equivalence relation because at least one of the properties weren’t met.

Let’s do the second row.

It is reflexive because A took 2 classes with A (one person shares the same two classes).

It is symmetrical because A took 2 classes with B, therefore B took 2 classes with A.

It is NOT transitive because A took 2 classes with B, and B took classes with C. Did A took 2 classes with C? You’re not sure because B could’ve taken those 2 classes with C in classes that don’t have A.

This is NOT an equivalence relation because at least one of the properties weren’t met.

Let’s do the third row.

It is reflexive because A has the same major as A (one person shares the same major).

It is symmetrical because if A has the same major as B, B also has the same major as A.

It is transitive because if A had the same major as B and B had the same major as C, A and C have the same major.

This IS an equivalence relation because all properties were met.

**14.** Suppose that

R1={(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)},

R2={(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)},

R3={(2,4),(4,2)} ,

R4={(1,2),(2,3),(3,4)},

R5={(1,1),(2,2),(3,3),(4,4)},

R6={(1,3),(1,4),(2,3),(2,4),(3,1),(3,4)},

Determine which of these statements are correct.

Check ALL correct answers below.

For starters, let’s just find what each relation demonstrates. Remember to test all possibilities.

First, starting R1.

It’s reflexive. Why? To be reflexive, every object must be related to itself (*x*, *x*). The pairs (2,2) and (3,3) exist.

It’s NOT symmetrical. Why? To be symmetrical, if (*x*, *y*), then (*y*, *x*) must exist. However, there’s missing corresponding pairs for (2,4) and (3,4).

It’s transitive. Why? To be transitive, (*x*, *y*) and (*y,* *z*) means that (*x*, *z*). When (2,3) and (3,4), there was also a relation (2,4). The same can be said for (3,2) and (2,4), in which the relation (3,4) exists.

Secondly, R2:

It’s reflexive. Pairs (1,1), (2,2), (3,3), and (4,4) exist.

It’s symmetrical. All pairs are reflections of one another.

It’s transitive. All pairs fit the principle.

Thirdly, R3.

It’s NOT reflexive. No pairs match with the principle.

It’s symmetrical. Both pairs are reflections.

It’s NOT transitive. There is no pair (2,2) in the relation.

Fourthly, R4.

It’s NOT reflexive. No pairs match with the principle (would need pairs (1,1), (2,2), (3,3)).

It’s NOT symmetrical. No pairs have reflections.

It’s NOT transitive. There’s no (1,3) or (2,4).

Fifthly, R5.

It’s reflexive. All pairs in the relation match with the principle.

It’s symmetrical. All pairs in the relation match with the principle.

It’s transitive. All pairs in the relation match with the principle.

Sixthly, R6.

It’s NOT reflexive. Each *x* value does not relate to itself.

It’s NOT symmetrical. Pairs missing are (4,1), (4,2), and (3,2).

It’s NOT transitive. Pairs missing are (4,1), any *x* = 4 pairs, (2,1).

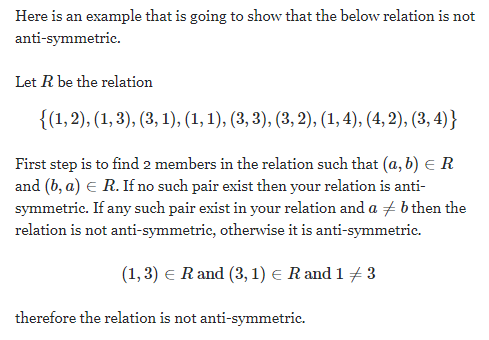
Lastly, one of the multiple choice options asks if R4 is antisymmetric.

This means that if relation pairs (*x*, *y*) and (*y*, *x*) exist, then *x* = *y*. In other words, *x* is related to *y* and *y* is related to *x* only if *x = y*.

Consider a relation to be antisymmetric UNLESS there is a counterexample, like unless there exists (*x*, *y*) and (*y*, *x*) where *x* is NOT equal to *y*. This goes for other properties as well; a property fails to hold IF AND ONLY IF a counterexample exists.

In the relation, there is no pair (*x*, *y*) that also has a (*y*, *x*) so it is asymmetric.

Good explanation about antisymmetric:



Let’s check the ones that are true.

**R4 is antisymmetric**

**CHECK AGAIN LATER**

**15.** Given the following relations on the set of all integers where (*x*, *y*) is *R* relation if and only if the following is satisfied. Check ALL correct answers from the following lists:

1. is an integer

Part A

Note: For 2 numbers to equal to 0, one must be a positive integer and the second must be a negative integer with the same absolute value. Or, they both must be 0.

Is it reflexive? No. Does *x + x =* 0? Only if *x* is 0, but you’re talking about ALL integers.

Is it irreflexive? Yes. *x* does not relate to itself.

Is it symmetric? Yes. Addition can be in any order, as long as other operations aren’t included.

Is it transitive? No. If *x + y* = 0 and *y + z* = 0, then *x + z* doesn’t equal 0, because then *x* = *z*.

Is it antisymmetric? No, because *x* and *y* must be different values in order to add up to 0.

Part B

What is an integer? It’s a whole number.

Is it reflexive? Yes, because always and 0 is always an integer.

Is it irreflexive? No because *x* is related to itself.

Is it symmetric? Yes, because if is an integer, is also an integer, because the negative of an integer is also an integer.

Is it transitive? Yes because if and are integers, then also equals an integer, because the difference of 2 integers is also an integer.

Is it antisymmetric? No, because if you insert values for *x* & *y* (like 1 and 2) for and , you get integer answers for both and .

Part C

Is it reflexive? No, because if you replace *y* with *x*, you get , and the only way for this to be true is if *x* is equal to 0 instead of a set of ALL integers like it says in the problem.

Is it irreflexive? Yes, because *x* is not related to itself.

Is it symmetric? No, because if you plug in values that make true and then put them in for , the equations won’t come out true.

Is it transitive? No, because if , then , which is not equal to .

Is it antisymmetric? Yes, because if and , then since can only be true when *x =* 0 (and therefore ).

Like in here, you should relate things to one another like in the transitive property.

Part D

Is it reflexive? No, because you can’t have (0,0). This is a sample value.

Is it irreflexive? Yes because *x* does not relate to itself.

Is it symmetric? Yes. Multiplication doesn’t have to be ordered.

Is it transitive? Yes because if *xy* > 1 and *yz* > 1, then *xz* > 1, given that they’re all nonzeroes with the same sign.

Is it antisymmetric? No. *xy* are related to each other both ways.

**CHECK LATER**

**16.** Given the following relations on the set of all people. Check ALL correct answers from the following lists:

1. *a* is older than *b*
2. *a* and *b* have a common grandparent
3. *a* has the same first name as *b*
4. *a* and *b* were born on the same day

Part A

Is it reflexive? No. If A is older than B, then A is older than A. This is false.

Is it irreflexive? Yes. “A is older than A” is false and does not relate to itself.

Basically means that if *x* is not related to itself, it’s irreflexive

Is it symmetric? No. If A is older than B, then B is older than A. This is false.

Is it transitive? Yes. If A is older than B and B is older than C, then A is older than C. It’s true.

You can assign ages to the variables. A is 70, B is 60, and C is 50. A is definitely older.

Is it antisymmetric? Yes. This is because B is not older than A.

Part B

Assume this means that *g* is a grandparent of both A and B. The term “common grandparent” does not mean the SAME.

Is it reflexive? Yes. A has a common grandparent with themselves.

Is it irreflexive? No because A and A are related to each other.

Is it symmetric? Yes. A has a common grandparent with B, and vice versa.

Is it transitive? No. *h* can be a grandparent of B and C, but *h* doesn’t have to be related to A.

Is it antisymmetric? No. With (*x*, *y*) also exists an (*y*, *x*)

Part C

Is it reflexive? Yes. A has the same name as A.

Is it irreflexive? No because A and A are related to each other.

Is it symmetric? Yes. A has the same name as B, and B has the same name as A.

Is it transitive? Yes. B has the same name as C, meaning A and C have the same name.

Is it antisymmetric? No. See reasoning for Part B.

Part D

Is it reflexive? Yes. A was born on the same day as A.

Is it irreflexive? No because A and A are related to each other.

Is it symmetric? Yes. A was born on the same day as B, and B was born on the same day as A.

Is it transitive? Yes. B was born on the same day as C, and A was born on the same day as C.

Is it antisymmetric? No. See reasoning for Part B.

**17.** Answer the following questions where *div* is for finding the integer quotient and *mod* is for remainder.

27 div 5 = **5**

27 mod 5 = **2**

-23 div 10 = **-3**

-23 mod 10 = **7**

21 div 7 = **3**

21 mod 7 = **0**

-22 div 10 = **-3**

-22 mod 10 = **8**

22 div 3 = **7**

22 mod 3 = **1**

-25 div 8 = **-4**

-25 mod 8 = **7**

397651648 mod 101 = **3**

801689212 mod 101 = **96**

The easiest way to figure out the mod is through the TI-89 calculator.

1. 2nd, 5 (for the MATH option)
2. Enter (the first option)
3. ALPHA = (which is the A)
4. Enter the first number FIRST
5. Comma the second smaller number

As for the *div* problems, input the expression into the calculator and use the whole number as your answer. If you want to think about doing the *mod* way this way too, just subtract the whole number you get from the first number to get the remainder.

If you want to do it by hand, you can solve them the traditional division way. Write out the numbers properly in a division “house”.

ONLY DO THIS WAY IF IT’S A POSITIVE DIVIDEND. THIS IS BECAUSE REMAINDERS MUST BE NON-NEGATIVE NUMBERS.

If it’s a negative dividend, ,

For -23 div 10, you can write it out like this:

where -3 was the div and 7 was the mod.

Remember, the remainder *must* always be positive. For example, you could put -2 as the div, but the remainder will be negative. So, you must add -1 to the div in order to get a positive remainder.

The other problems look like this:

and

**18.** Perform the following congruence computations. Make sure that the number you enter is and , where *N* is the modulus of the congruence.

[What does mean?](https://www.youtube.com/watch?v=6dZLq77gSGU)

1. *a* and *b* have the same “remainder” when they are divided by *n*
2. , where *k* is some integer you don’t know yet (This is NOT remainder equation in notes)
3. , in which the | means “divides”, so is a multiple of *n*

You can use these to check if your answer is correct. Using this, you can also figure out the answer to each question.

Part A

First, add the left side up to get 9749 as the total.

Next, take the remainder of using the TI-89 calculator to get **38** as the remainder.

Why is this the answer? You can check using the statements above.

This is because first, if you used any other number, the answer would be incorrect because it would be greater than *n* (or the value enclosed in parentheses next to mod).

Secondly, the remainder of will be 38. This fulfills the first statement above. Both remainders become 38.

Thirdly, if you plug in the values for the equation, you can prove that both the second and third statement work. *k* comes out to an integer when you solve for it, and 9711 (the expression ) is a multiple of *n* because of that.

However, notice that *b* can always be smaller or greater. As long as *b* has the same remainder as *a*, any value can be plugged in for *b*, because the sign in between is equivalent.

Part B

First, combine the left side using PEMDAS. You get .

Second, you can write out an equation with this, like where *k* is some integer. Since you’re finding the remainder for a negative dividend, this value, *k* must be a negative integer. To find a perfect value for this, do to get around -1303 for *k*. Increase the absolute value of it by 1 (so you get -1304 for *k*) so that you get a positive integer in the end when solving for *r*, the remainder.

So, 97 times -1304 is -126488. So, solve as you normally would in algebra, and you get *r* = **26** as the remainder. You can test this out as well.

Part C

Solve as much as you can on both sides. You get . Use the first principle stated underneath the video to get 8 as the remainder for both and 172568.

*My way (less safe)*

So, since it is the remainder, you can create a new equation: . This certainly makes the numbers smaller and easier to deal with.

Divide 18681 by 60 to get the actual number 311.35. Round and input this as *k* in the equation and then add 8. You get . You can subtract the 18681 on both sides and you get , which is the correct answer.

*Other way*

Find the remainder of 18681 and 60. The remainder of this is 21. The equation then can be written out as . Plug in values for *m* that would make the remainder of the left side to be also 8. Guess and check.

Or, algebraically, subtract the 18681 on both sides and find the remainder of that (-18673). Plug it into the division theorem, find a good integer for *q* that makes *nq* just over -18673 (this is 312, like in the less safe way).

47, 27, 25

Part D

Solve as much as you can on both sides. So, you get . Use the TI-89 to find the remainder and that is the answer (**27**). This is because of the first principle stated above. *a* and *b* must have the same remainder. Since the remainder is lesser than 44, automatically the remainder of *b* is itself in this case.

**FILL IN LATER**

Part E

Solve as much as you can on both sides. So, you get . Solve the same way as Part C and you get **25**.

**19.** The remainder when *a* is divided by 7 is 2 and the remainder when *b* is divided by 7 is 5, so . Find:

Hint: Test some specific values for *a* and *b* that satisfy the hypotheses.

First, the hypotheses refer to the mod equations in the problem. Find some specific values for a and b, like the hint says. How?

Let’s do .

This basically equals . To create a sample value (that you can plug in to do the different parts), you can replace the quotient with any number. Choosing 1 is best because it gives the smallest number *a* can be. So, for this, .

Let’s do .

This basically equals . Do the same thing as done with *a*. So, for this, .

For each part, plug in values for *a* and *b* and solve like in 17.

1. **4**
2. **0**
3. **6**
4. **3**

**20.** Find the smallest positive integer for which and . What is the next smallest integer with this property? (You will have to do some trial and error, but thinking about divisibility should lead you to some patterns.)

According to ["Everything You Need to Know About Modular Arithmetic"](http://pi.math.cornell.edu/~morris/135/mod.pdf), let *m* > 0 be a positive integer called the modulus. Two integers *a* and *b* are congruent modulo *m* if is divisible by *m*. In other words (check the PDF for some notes about the equation),

for some integer *k*

Smallest positive integer

Create a system of equations for both equations: and . These are written in the same format as Euclid’s Division Theorem. Since they’re both equal to each other, set them that way, then solve for *k*.

Why? Because that way, when you solve for *k*, the coefficient for *q* is larger than the coefficient of *k*, which is essential to solve this problem.

You should get . You can simplify this to because you can split 4*q* into 3*q + q*, then split the fraction and put them each over 3. The 3*q* simplifies to *q* on the outside of the fraction. By some divisibility rule (not sure how or what at the moment), you can set the numerator equal to the denominator ONLY when the variable in the numerator has a coefficient of 1.

So, you can set , and you’ll get . So, you can insert this in the original equation and get **11** as the highest positive integer.

To check, you can plug in *q* to the *k =* equation and get *k* as 3, and you can plug that *k* value into the equation to also get 11.

I tried this by making a similar problem and it also worked for that too.

Next smallest integer

You could use this to solve the first part of this problem too.

First, notice here that *x* is equivalent to *q* in the equation , in which *q* is equivalent to 2 (as found in the first part). Remember, “equivalent” does not mean “equal”, it means that one of *q*’s values *can* be 2. So, you get

And, for some integer *k*, you can plug in the expression (on the right side) for *q* and combine it like so:

You can plug in any value for *k*. For the first smallest integer, you can plug in 0 for *k*, and you get the 11 (the answer to the first problem). Plug in 1 for *k* to get the next smallest digit which is **23**.

**MIGHT NEED MORE EXPLANATION**